

Exercise 2.5: fractal boundaries

In the book *Understanding Nonlinear Dynamics* by D. Kaplan and L. Glass (Springer, New York, 1995), the authors propose the study of the following map:

$$x_{t+1} = (x_t + a + b \sin 2\pi x_t) \pmod{1} \quad (1)$$

Here, the “Modulus” operator takes the fractional part of a number. For instance, the modulus of 4.28, that is $4.28 \pmod{1}$ is 0.28.

Exercise: Study the map starting from different initial conditions.

Solution: We use the following code, in which $a = 0.53$ and $b = 0.62$, as in D. Kaplan and L. Glass. The initial condition x_0 (in the code `xinit`) is equal to 0.06 and 0.07.

```
# Basin1: cobweb plots with two initial conditions
f.x<-function(x,a,b,pi2){ (x+a+b*sin(pi2*x))%1}
pi2<-2*pi
##### function to draw the cobweb plot #####
iter<- function(xinit,nstep,a,b,pi2){      # starting function iter
x<- xinit
y<- f.x(x,a,b,pi2)
if(xinit ==xinit1){segments(x,0,x,y,lty=4,lwd=2,col="green3")}
if(xinit ==xinit1){points(x,y,pch=19, col = "green3", cex=1.0)}
if(xinit ==xinit2){segments(x,0,x,y,lty=1,lwd=2,col="red")}
if(xinit ==xinit2){points(x,y,pch=22, col = "red", cex=1.5)}
for(i in 1:nstep){
if(xinit ==xinit1){segments(x,y,y,y,lty=4,lwd=2,col="green3")}
if(xinit ==xinit2){segments(x,y,y,y,lty=1,lwd=2,col="red")}
x<- y
y<- f.x(x,a,b,pi2)
if(xinit ==xinit1){segments(x,x,x,y,lty=4,lwd=2,col="green3")}
if(xinit ==xinit1){points(x,y,pch=19, col = "green3", cex=1.0)}
if(xinit ==xinit2){segments(x,x,x,y,lty=1,lwd=2,col="red")}
if(xinit ==xinit2){points(x,y,pch=22, col = "red", cex=1.5)}
}
}      # ending function iter

### parameters and initial conditions
a<- 0.53
b<- 0.62
nstep<- 10
xinit1<- 0.06
xinit2<- 0.07

### ### preparation of the cobweb plot
plot(0,0,type="n",xlim=c(0,1),ylim=c(0,1),xlab="x(t)",ylab="x(t+1)",
```

```

cex.lab=1.5,cex.axis=1.2)
segments(0,0,1,1,lty=3,lwd=2,col="magenta") # bisector
curve(f.x(x,a,b,pi2),from = 0,      to = 0.1, lty=5,lwd=2,col="blue",add=T)
curve(f.x(x,a,b,pi2),from = 0.102,  to = 0.51,lty=5,lwd=2,col="blue",add=T)
curve(f.x(x,a,b,pi2),from = 0.52,    to = 0.88,lty=5,lwd=2,col="blue",add=T)
curve(f.x(x,a,b,pi2),from = 0.90,    to = 1,   lty=5,lwd=2,col="blue",add=T)

xinit<- xinit1          # first initial condition
iter(xinit,nstep,a,b,pi2) # call up the cobweb plot
xinit<- xinit2          # second initial condition
iter(xinit,nstep,a,b,pi2) # call up the cobweb plot

```

The result is shown in Fig. 1. We are in presence of two stable periodic cycles and so we speak of *multistability*. If $x_{\text{init}} = 0.06$ there is a 2-period cycle, with fixed points $x_t = 0.6773$ and $x_{t+1} = 0.650867$. If $x_{\text{init}} = 0.07$ there is a 4-period cycle, with fixed points $x_t = 0.39848$, $x_{t+1} = 0.29765$, $x_{t+2} = 0.42007$, $x_{t+3} = 0.24852$.

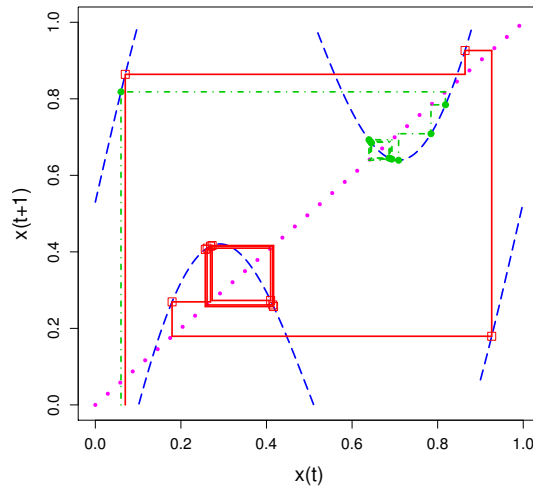


Figure 1 Cobweb plot of the map (1) with $a = 0.53$ and $b = 0.62$. With the initial condition $x_{\text{init}} = 0.06$, the iterates converge to a 2-period cycle (dot-dash line), while with $x_{\text{init}} = 0.07$, the convergence is toward a 4-period cycles (solid line). The (1) is plotted with dashed lines.

Let x_0 be in the whole interval $[0, 1]$. There are only two possible behaviors for the map: the iterates either approach the 2-period cycles, or approach the 4-period cycles. With the following code, a black vertical segment is plotted when the iterates converge to the 2-period cycle, while a gray vertical segment is plotted when the iterates converge to the 4-period cycle. Our intention is to distinguish the two basins of attraction, that is to identify the

boundaries of each basin. The following code teaches us that the task is not so clear-cut.

```
#Basin2: fractal boundaries
f.x<-function(x,a,b,pi2){(x+a+b*sin(pi2*x))%1}
pi2<-2*pi
a<- 0.53
b<- 0.62
ntrans<- 500      # transient
n<-50             # number of iterations after the transient
nt<-ntrans+n      # total iterations
nr<-1000          # number of initial conditions in [0,1]
xinit<- 0
xfin <- 1
pa<-(xfin-xinit)/nr # length of the step from 0 to 1
xinit<- xinit-pa
yv<- 0.5          # length of vertical segments
par(mfrow=c(3,1)) # we will create a multi-paneled plot with 3 rows and 1 column
plot(1,1,type="n",xlim=c(0,1),ylim=c(0,yv),xlab="",ylab="",yaxt="n",
     cex.lab=1.8,cex.axis=1.5)
for(x in 1:nr) {
  xinit<-xinit+pa
  x<-xinit
  for (j in 1:nt) {
    y<- f.x(x,a,b,pi2)
    if(j>ntrans){
      if((x>0.297) & (x<0.298)) segments(xinit,0,xinit,yv,lwd=1,col="gray")
      if((x>0.650) & (x<0.651)) segments(xinit,0,xinit,yv,lwd=1,col="black")
      x<-y
    }
  }
}

##### first zooming-in

xinit<- 0.
xfin <- 0.1
nr<-5000
pa<-(xfin-xinit)/nr
xinit<-xinit-pa
yv<- 0.5
plot(1,1,type="n",xlim=c(xinit,xfin),ylim=c(0,yv),xlab="",ylab="",yaxt="n",
     cex.lab=1.8,cex.axis=1.5)
for(x in 1:nr) {
  xinit<-xinit+pa
  x<-xinit
```

```

        for (j in 1:nt) {
            y<-f.x(x,a,b,pi2)
        if(j>ntrans){
        if((x>0.297) & (x<0.298)) segments(xinit,0,xinit,yv,lwd=1,col="gray")
        if((x>0.650) & (x<0.651)) segments(xinit,0,xinit,yv,lwd=1,col="black") }
            x<-y
        }
    }

##### second zooming-in
xinit<- 0.06
xfin <- 0.08
pa<-(xfin-xinit)/nr
xinit<-xinit-pa
yv<- 0.5
plot(1,1,type="n",xlim=c(xinit,xfin),ylim=c(0,yv),xlab="x(t)",ylab="",yaxt="n",
cex.lab=1.8,cex.axis=1.5)
for(x in 1:nr) {
xinit<-xinit+pa
x<-xinit
    for (j in 1:nt) {
        y<- f.x(x,a,b,pi2)
    if(j>ntrans){
    if((x>0.297) & (x<0.298)) segments(xinit,0,xinit,yv,lwd=1,col="gray")
    if((x>0.650) & (x<0.651)) segments(xinit,0,xinit,yv,lwd=1,col="black") }
        x<-y
    }
}

```

Figure 2 shows that when we start with $x_0 \in [0, 1]$ and we see that there are large enough regions in which two nearby initial conditions lead to the same fixed point. But if we zoom in on the small interval $x_0 \in [0, 0.1]$, we see that the region $[0, 0.1]$ which appeared almost entirely black contains in fact smaller gray zones. The same result happens if we zoom in on the even smaller region $[0.06, 0.08]$. The conclusion is that we cannot trace a line delimiting the boundaries of the basins, but they are interleaved in a fractal way.

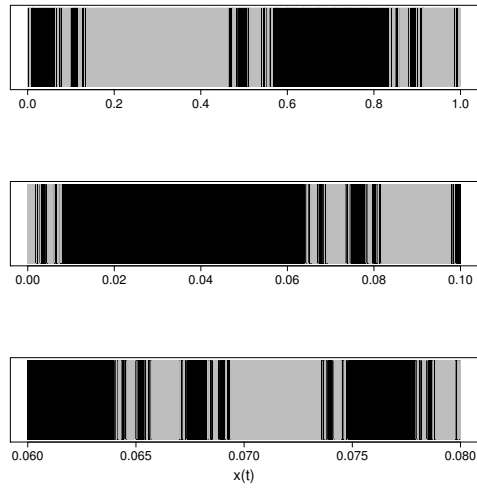


Figure 2 On the axes of abscissae are reported the intervals of the initial conditions from which of the iterates of the map (1) ($a = 0.53$ and $b = 0.62$) start. Top: $[0, 1]$, middle: $[0, 0.1]$, bottom: $[0.06, 0.08]$. A black vertical segment is plotted when the iterates converge to the 2-period cycle and a gray vertical segment is plotted when the iterates converge to the 4-period cycle.