

Exercise 7: some pendulum

In the exercises for the Chapter 6, *Searching for fixed points*, we study the solutions of linear systems of equations. Here we propose some exercises regarding *nonlinear* systems. From an educational point of view, these exercises should follow those for the Chapter 6, but since they regard pendula systems they allow us to extend the ideal simple pendulum model (with small oscillations) introduced in Sect. 3.2

Ideal pendulum with unlimited oscillations

If the energy of the pendulum is greater than a critical value ($E > 2mgl$) the pendulum is able to spin around in a full circle. This motion is called *rotation*. In the case in which the energy is not enough ($0 < E < 2mgl$), the pendulum oscillates from right to left and vice versa. This motion is called *libration*, from the latin *libra* (*scales*). Notice that also the rotation, as the libration, is not uniform. The angular velocity is at its maximum, when the bob is in the lowest point (position of stable equilibrium), and is at its minimum, when the bob is in its highest point (position of unstable equilibrium).

The angles are no longer “small”, so eqn (3.2) becomes

$$\frac{d^2x}{dt^2} + \omega^2 \sin x = 0$$

or equivalently:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\omega^2 \sin x \end{cases}$$

Exercise: Derive the critical value $2mgl$.

Solution: Since the pendulum is “ideal”, there are no friction or air resistance, so the total mechanical energy E of the system is conserved, that is the sum of its kinetic energy E_k and potential energy E_p remains constant during the motion. Such systems are called *conservative*. Since a mass m rotating on a circular trajectory of ray l , has speed $v = l(d\theta/dt)$, the kinetic energy is given by:

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}ml^2 \left(\frac{d\theta}{dt} \right)^2$$

For the potential energy E_p , see Fig. 3.1. When the bob is at the bottom, that is $\theta = 0$, we assume the potential energy to be zero. When the bob is at a height h , swinging by an angle θ , it is:

$$E_p = mgh = -mg(l - l \cos \theta) = mgl(1 - \cos \theta)$$

then:

$$E = E_k + E_p = \frac{1}{2}ml^2 \left(\frac{d\theta}{dt} \right)^2 + mgl(1 - \cos \theta) \quad (1)$$

also:

$$\frac{1}{2}ml^2 \left(\frac{d\theta}{dt} \right)^2 = E - mgl(1 - \cos \theta)$$

It follows:

$$\frac{d\theta}{dt} = \pm \sqrt{\frac{2}{ml^2} (E - mgl(1 - \cos \theta))}$$

It must be:

$$E - mgl(1 - \cos \theta) > 0$$

that is:

$$mgl(1 - \cos \theta) > mgl - E \quad \text{or} \quad \cos \theta > 1 - \frac{E}{mgl} \quad (2)$$

We are in presence of two conditions. If

$$1 - \frac{E}{mgl} < -1 \quad \text{that is} \quad E > 2mgl$$

there are no problems, eqn (2) holds for any value of θ and the motion is an illimitate motion of rotation. Otherwise:

$$1 - \frac{E}{mgl} > -1 \quad \text{that is} \quad 0 < E < 2mgl$$

then the solution of eqn (2) exists only with a certain interval $\theta_1 < \theta < \theta_2$ and the motion is a limited motion of libration.

Since

$$1 - \cos \theta = 2 \sin^2 \left(\frac{\theta}{2} \right)$$

eqn (1) can be rewritten as:

$$E = E_k + E_p = \frac{1}{2}ml^2 \left(\frac{d\theta}{dt} \right)^2 + 2mgl \sin^2 \left(\frac{\theta}{2} \right)$$

being $\omega^2 = g/l$

$$E = \frac{1}{2}ml^2 \left[\left(\frac{d\theta}{dt} \right)^2 + 4\omega^2 \sin^2 \left(\frac{\theta}{2} \right) \right] \quad (3)$$

To determine the kind of motion, we have to see if $E > 2mgl$ or $E < 2mgl$. Now, $2mgl = 2m\omega^2 l^2$, so we have to see if

$$\left(\frac{d\theta}{dt} \right)^2 + 4\omega^2 \sin^2 \left(\frac{\theta}{2} \right) > 4\omega^2 \quad \text{or} \quad < 4\omega^2 \quad (4)$$

Exercise: Find the fixed points and discuss their stability (instability).

Solution: To find the fixed points, we solve the system:

$$\begin{cases} \frac{dx}{dt} = y & = 0 \\ \frac{dy}{dt} = -\omega^2 \sin x & = 0 \end{cases}$$

There is the solution $(x^*, y^*) = (0, 0)$, but also there are solutions for those x values such that $\sin x = 0$, that is the solutions are given by:

$$\dots, (-2\pi, 0), (-\pi, 0), (0, 0), (+\pi, 0), (+2\pi, 0), \dots$$

In general, the fixed points are in:

$$(x^*, y^*) = (n\pi, 0), n = \dots, -2, -1, 0, +1, +2, \dots$$

including the solution $(0, 0)$. Let us see the stability (instability) of these points. As we have said for the logistic map (Sect. 2.4), we have to linearize the system of equations around the fixed points.

We known that the linear system is given by:

$$\begin{cases} \frac{dx}{dt} = f(x, y) = ax + by \\ \frac{dy}{dt} = g(x, y) = cx + dy \end{cases}$$

where a, b, c, d are not all zero. If the system is not linear and it is linearized around the fixed point (x^*, y^*) , let $z = x - x^*$ and $w = y - y^*$, it becomes:

$$\begin{cases} \frac{dz}{dt} = Az + Bw \\ \frac{dw}{dt} = Cz + Dw \end{cases}$$

where:

$$\begin{cases} A = \left. \frac{\partial f(x, y)}{\partial x} \right|_{(x^*, y^*)} & B = \left. \frac{\partial f(x, y)}{\partial y} \right|_{(x^*, y^*)} \\ C = \left. \frac{\partial g(x, y)}{\partial x} \right|_{(x^*, y^*)} & D = \left. \frac{\partial g(x, y)}{\partial y} \right|_{(x^*, y^*)} \end{cases}$$

In our case:

$$\begin{cases} \frac{dx}{dt} = f(x, y) = y \\ \frac{dy}{dt} = g(x, y) = -\omega^2 \sin x \end{cases}$$

therefore:

$$\begin{cases} A = 0 & B = 1 \\ C = -\omega^2 \cos x|_{(x^*=n\pi)} = \begin{cases} -\omega^2 & (n \text{ even}) \\ +\omega^2 & (n \text{ odd}) \end{cases} & D = 0 \end{cases}$$

Then:

$$\begin{cases} \frac{dz}{dt} = w \\ \frac{dw}{dt} = \pm \omega^2 z \end{cases}$$

These systems of equations are solved through the *characteristic equation*, as explained in Appendix C.

$$\lambda^2 - (a + d)\lambda - (ad - bc) = 0$$

The eigenvalues are the roots of this equation:

$$\begin{aligned}\lambda_1 &= \frac{a + d}{2} + \frac{\sqrt{(a - d)^2 + 4bc}}{2} \\ \lambda_2 &= \frac{a + d}{2} - \frac{\sqrt{(a - d)^2 + 4bc}}{2}\end{aligned}$$

In our case, the characteristic equation is given by:

$$\begin{aligned}\lambda^2 - (-\omega^2) &= 0 \text{ (} n \text{ even)} \\ \lambda^2 - (+\omega^2) &= 0 \text{ (} n \text{ odd)}\end{aligned}$$

As discussed in Sect. 6.2, for the “even” points, $\lambda^2 = -\omega^2$, the eigenvalues are $\lambda_1 = -i\omega$ and $\lambda_2 = +i\omega$, so the “even” points are centres, nonhyperbolic equilibrium points. The “odd” points have the eigenvalues $\lambda_1 = -\omega$ and $\lambda_2 = +\omega$, so they are saddles, hyperbolic equilibrium points. The following code depicts the pendulum motions in the phase space. Remember: x is the angle θ , y is the angular velocity $d\theta/dt$. The code is Code 6.1, based on R package `phaseR` (Grayling, 2015), with some small changes to vary the initial conditions (`xy0`) and ω^2 (`omega2`). Only the trajectories in the phase space are plotted (`trajectory<-trajectory(. )`), while the nullclines (see Section 6.2) are not computed.

```
# Ideal pendulum: unlimited oscillations (pend_id_un_o)
library(phaseR)
#Set parameters for integration
t.end<- 20 #integrate from t=0 to t.end
delta<-0.05

pend<-function(t,y,parameters){
  x <- y[1]
  y <- y[2]
  a<-parameters[1]
  b<-parameters[2]
  c<-parameters[3]
  d<-parameters[4]
  dy<-numeric(2)
  dy[1]<- a*x+b*y
  dy[2]<- c*sin(x)+d*y
  list(dy)
}

# ROTATION

omega2<- 1.
parms<-c(0,1,-omega2,0) #parameter values
x0<-c(-4*pi,4*pi)
y0<-c(4,-4)
xy0<-cbind(x0,y0) #initial conditions
```

```

plot(0,0,type="n" ,xlim=c(-4.5*pi,4.5*pi),ylim=c(-5,5),
cex.axis=1.2,cex.lab=1.5,font.lab=3,xlab="x",ylab="y")
trajectory<-trajectory(pend,y0=xy0,t.end=t.end,t.step=delta,parameters=parms,
lwd=3,lty=2,colour=rep("blue",2),add=TRUE)

omega2<- 1.01
parms<-c(0,1,-omega2,0) #parameter values
delta<-0.05
x0<-c(-4*pi,4*pi)
y0<-c(2.2,-2.2)
xy0<-cbind(x0,y0) #initial conditions
trajectory<-trajectory(pend,y0=xy0,t.end=t.end,t.step=delta,parameters=parms,
lwd=3,lty=2,colour=rep("blue",2),add=TRUE)

# LIBRATION

omega2<- 2.5
parms<-c(0,1,-omega2,0) #parameter values
t.end<- 100
delta<-0.05
x0<-c(1-4*pi,1-2*pi,1,1+2*pi,1+4*pi)
y0<-c(1,1,1,1,1)
xy0<-cbind(x0,y0) #initial conditions
trajectory<-trajectory(pend,y0=xy0,t.end=t.end,t.step=delta,parameters=parms,
lwd=2,lty=1,colour=rep("black",5),add=TRUE)
omega2<- 0.5
parms<-c(0,1,-omega2,0) #parameter values
t.end<- 20
delta<-0.05
x0<-c(1-4*pi,1-2*pi,1,1+2*pi,1+4*pi)
y0<-c(1,1,1,1,1)
xy0<-cbind(x0,y0) #initial conditions
trajectory<-trajectory(pend,y0=xy0,t.end=t.end,t.step=delta,parameters=parms,
lwd=2,lty=1,colour=rep("black",5),add=TRUE)

# SEPARATRIX

omega2<- 1
parms<-c(0,1,-omega2,0) #parameter values
t.end<- 100
delta<-0.05
x0<-c(-5*pi,-3*pi,-pi,pi,3*pi,5*pi)
y0<-c(0.001,0.001,0.001,-0.001,-0.001,-0.001)
xy0<-cbind(x0,y0) #initial conditions
trajectory<-trajectory(pend,y0=xy0,t.end=t.end,t.step=delta,parameters=parms,
lwd=2,lty=3,colour=rep("red",2),add=TRUE)

seq_pie<-seq(-4*pi,4*pi,by=2*pi) # even
seq_pie
points(seq_pie,rep(0,5),col="black",pch=19,cex=0.8)
seq_pio<-seq(-5*pi,5*pi,by=2*pi) # odd
seq_pio
segments(seq_pio,-5,seq_pio,5,col="green3",lwd=2,lty=4)

```

Figure 1 shows the result. From an overall view, it appears at once the periodicity in x

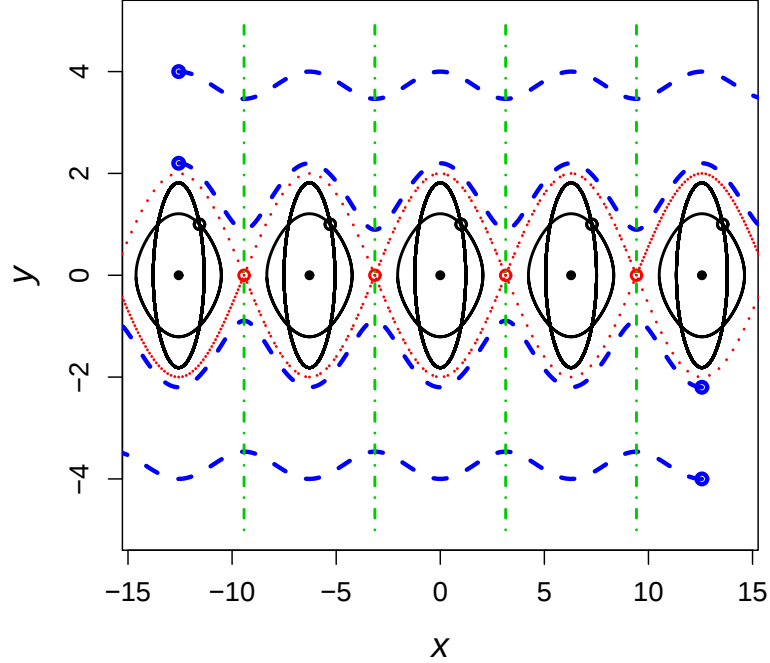


Figure 1 Phase portrait of some trajectories of the ideal pendulum with unlimited oscillations. The solid curves refer to libration motions, the dashed curves to rotation motions, the dotted curves represent separatrices; centres are plotted as black dots, dash-dot lines cross the horizontal axis at saddle points. The small circles indicate the starting points.

and the symmetry around the x and y axes. In fact, no physical difference exists between states with angles that differ by 2π and between states (x, y) and $(-x, -y)$. As we found above, the points at $(2n\pi, 0)$ are centres, if the pendulum is in the state $(0, 0)$, it is at rest, remaining there forever. Also the points at $((2n + 1)\pi, 0)$ are equilibrium points, but with unstable equilibrium (saddles). In these points the pendulum is at the inverted position, the bob is standing at the top, and being the pendulum ideal, it will stay there forever.

If a tiny perturbation is applied to the bob, from both equilibrium states the pendulum begins to swing. If it is slightly pushed from its stable equilibrium point (at rest at the bottom) the trajectories are like those we have seen for the linear system, that is nearly ellipses (see Fig. 3.2). Mathematically, it means that $E < 2mgl$. Figure 1 shows two of such trajectories with continuous lines. To show the periodicity in x , the initial points are chosen with the same y , and at even multiples of π . Physically, it means that the initial angular velocities $d\theta/dt$ are all the same, while the initial angles θ vary periodically with period 2π . In other words, the angles θ and $\theta + 2\pi, \theta + 4\pi, \dots$, and also $\dots, \theta - 4\pi, \theta - 2\pi$ are equivalent, indicating the same position of the bob. As a consequence, it is sufficient to study the dynamics of the system in the interval $[-\pi, +\pi]$, as we have done in Sect. 4.5, where the damped driven pendulum is discussed. We have seen that when the trajectory crosses the left boundary $-\pi$, it is translated from left to right into $[-\pi, +\pi]$. The parameter ω^2 (omega2) is equal to 2.5 and 0.5. In the latter case, the ellipsis is nearer to a circle.

If we push the bob with more thrust ($E > 2mgl$), the system assumes a rotation motion. In this case the angular velocity is never zero, neither for $x = 0$, nor for $x = \pm\pi$. In the phase space, this behaviour is depicted by dashed lines, corresponding to different initial conditions and different ω^2 . For $y > 0$, the direction is to the right, while for $y < 0$, it is to the left (see also the small circles indicating the starting points). There is no physical difference if the pendulum rotates in the clockwise direction or in the counterclockwise direction, this means that the motion of the pendulum is *reversible*, property holding in general for conservative systems. Imagine to be in the state $(-3\pi, 3.4641)$, if the sign of the angular velocity is instantaneously changed, the system jumps in the state $(-3\pi, -3.4641)$, symmetric to the former with respect to the x -axis. All the states visited in the left-to-right motion are visited with the same angular velocity in the right-to-left motion. From this property a new symmetry emerges, called *symmetry of time reversal*. Indeed, changing the sign of the angular velocity ($y \rightarrow -y$), from a physical point of view is equivalent to change the sign of time ($t \rightarrow -t$).

Let us turn to the pendulum at the unstable equilibrium point $(0, \pi)$. In this position, the energy is exclusively potential energy equal to $2mgl$. Suppose to give an infinitesimal blow, without changing the energy. The pendulum will begin to swing, and it will return to the top and stop there. But it would be necessary an infinite time to complete such a trajectory which starts from a saddle point and returns to a saddle point. This type of trajectories are the separatrices represented in Fig. 1 with dotted lines. The separatrix is a pure ideal trajectory of an ideal pendulum.

We can write the energy E of the system (eqn (3)) adding the lines

```
Emec<- y0[k]^2 + 4*omega2*(sin(x0[k]/2))^2
Emec
```

with k equal to 1, 2. Thus we can check if $E < 2mgl$ or $E > 2mgl$ (eqn (4)). In the “separatrix condition”, with $\omega^2 = 1$ and with the initial conditions reported in the code, it results $\text{Emec} = 4.0000$, as it must be. Note that the values of the initial angles $x0$ are the x -coordinate of the saddle points, while the absolute values of the initial angular velocities $y0$ are all the same and very near to zero.

If the nullclines along with the velocity field are wanted, we have to add the functions **flowField** and **nullclines**. Given the linear system $dx/dt = f(x, y)$, $dy/dt = g(x, y)$, recall (Sect. 6.2) that the x -nullcline is the set of points in the phase plane where $dx/dt = 0$. Geometrically, these are the points where the vectors are vertical, going either upward or downward. Similarly, the y -nullcline is a set of points where $dy/dt = 0$. Geometrically, these are the points where the vectors are horizontal, going either to the left or to the right. At fixed points, the nullclines intersect, that is where $f(x, y) = g(x, y) = 0$.

With the function **stability**, we can check the type of equilibrium points. Both the x and y variables are plotted as a function of time, with the function **numericalSolution**. The code is reported below, one trajectory of libration motion is also computed.

```
# Ideal pendulum: unlimited oscillations (pend_id_un_o_1).
# trajectory of libration motion are plotted;
# nullclines and velocity field are plotted;
# equilibrium points are classified;
# x and y variables are plotted as a function of the time.
```

```

library(phaser)
pend<-function(t,y,parameters){
  x <- y[1]
  y <- y[2]
  a<-parameters[1]
  b<-parameters[2]
  c<-parameters[3]
  d<-parameters[4]
  dy<-numeric(2)
  dy[1]<- a*x+b*y
  dy[2]<- c*sin(x)+d*y
  list(dy)
}

# LIBRATION

omega2<- 2.5
parms<-c(0,1,-omega2,0) #parameter values
t.end<- 100
delta<-0.05
x0<- 0
y0<- 2
xy0<-cbind(x0,y0) #initial conditions
plot(0,0,type="n",xlim=c(-2.5*pi,2.5*pi),ylim=c(-10,10),
     cex.axis=1.2,cex.lab=1.5,font.lab=3,xlab="x",ylab="y")
trajectory<-trajectory(pend,y0=xy0,t.end=t.end,t.step=delta,parameters=parms,
                       lwd=2,lty=1,colour="black",add=TRUE)

arrows<-flowField(pend,lwd=2,x.lim=c(-2.5*pi,2.5*pi),y.lim=c(-10,10),
                  parameters=parms,points=19,arrow.head=0.05,frac=1,colour="gray60",add=TRUE)
null<-nullclines(pend,parameters=parms,lwd=2,points=151,
                 x.lim=c(-2.5*pi,2.5*pi),y.lim=c(-10,10),colour=c("red","blue"))

stability.1 <- stability(pend, y.star=c(0,0),parameters=parms)
stability.2 <- stability(pend, y.star=c(pi/2,0),parameters=parms)

windows()
numericalSolution<- numericalSolution(pend,y0=c(1,1),t.end=20,type="one",t.step=delta,
                                     parameters=parms,colour=c("green","orange"),lty=c(1,2),
                                     ylab="x, y",ylim=c(-2.5,2.5),lwd=2,cex.axis=1.2,cex.lab=1.5)
legend("bottomright",col=c("green","orange"),legend=c("x","y"),lty=c(1,2))

```

Figure 2 shows the result. Figure 3 shows the angle θ (x) and the angular velocity $d\theta/dt$ (y) as a function of time.

The reader is encouraged to run the code with different initial conditions and different values of ω^2 , keeping in mind that the dynamics of the system depends on all these parameters.

Damped pendulum

In Sect. 6.2 we have studied the behaviour of two-dimensional linear ODE systems. We have seen different equilibria points (stable or unstable): nodes, foci, centres, saddles. Here we propose some exercises in which the concepts of Sect. 6.2 are applied to the study of the dynamics behaviour of a *real* pendulum, “real” in that the damping of the oscillations

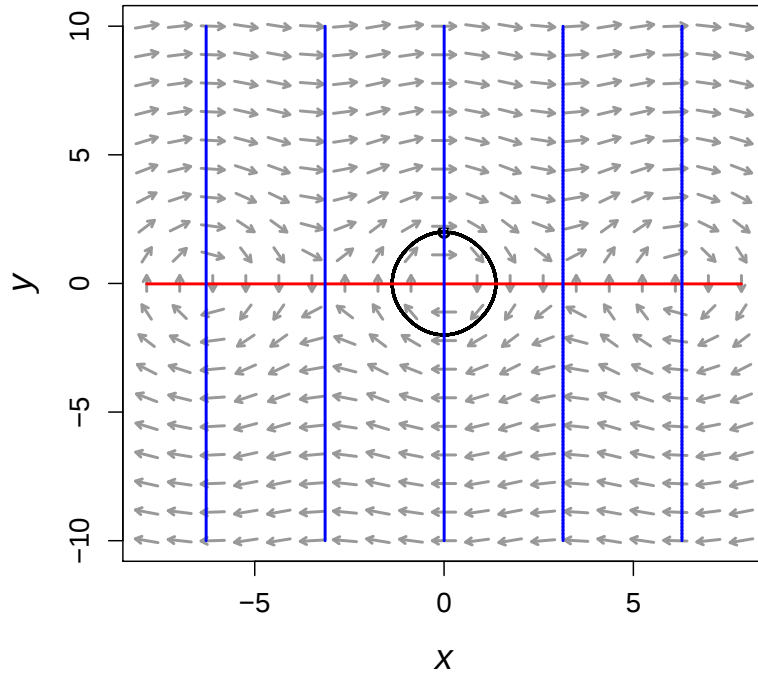


Figure 2 Phase portrait of one trajectory of the ideal pendulum with unlimited oscillations in the libration regime (solid curve). The nullclines and the velocity field are also plotted.

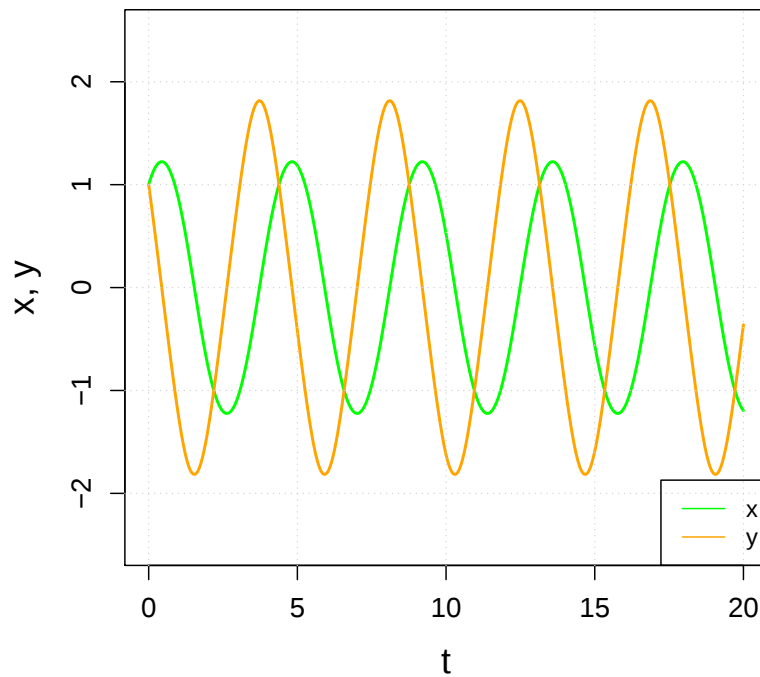


Figure 3 The angle θ (x) and the angular velocity $d\theta/dt$ (y) as a function of time for one trajectory of the ideal pendulum with unlimited oscillations in the libration regime.

(air and/or suspension point friction) is taken into account (see also Sect. 4.5). Consider the equation:

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega^2 x = 0$$

or equivalently the system:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\omega^2 x - \gamma y \end{cases} \quad (5)$$

which are as the equations (3.1) and (3.2), that is the ideal pendulum with small oscillations, but with the new term γy referring to the damping, called *damping constant*. In general, damping forces are very complicated, here we take a very simple one, that is proportional to angular velocity.

Exercise: Study the dynamical behaviour of the damped pendulum with small oscillations.

Solution: To write the characteristic equation

$$\lambda^2 - (a + d)\lambda - (ad - bc) = 0$$

consider that:

$$\begin{aligned} a &= 0 & b &= 1 \\ c &= -\omega^2 & d &= -\gamma \end{aligned}$$

so the characteristic equation is:

$$\lambda^2 + \gamma\lambda + \omega^2 = 0$$

The general solution is:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad (6)$$

with:

$$\lambda_{1,2} = \frac{1}{2} \left[-\gamma \pm \sqrt{\gamma^2 - 4\omega^2} \right]$$

Let us consider the square root.

$$\begin{aligned} \gamma^2 < 4\omega^2 &\longrightarrow \lambda_1, \lambda_2 \text{ complex conjugate} \\ \gamma^2 > 4\omega^2 &\longrightarrow \lambda_1, \lambda_2 \text{ both } < 0 \end{aligned}$$

Let us see the first case: $\gamma^2 < 4\omega^2$.

Both λ_1 and λ_2 are complex number of the type $a \pm ib$. Now:

$$e^{a+ib} = e^a (\cos b + i \sin b)$$

We can also write:

$$\begin{aligned} \cos \omega t &= \frac{1}{2} (e^{i\omega t} + e^{-i\omega t}) \\ \sin \omega t &= \frac{1}{2} (e^{i\omega t} - e^{-i\omega t}) \end{aligned}$$

Calling $\alpha = \sqrt{\omega^2 - \gamma^2}$, the general solution eqn (6) becomes:

$$x(t) = c_1 e^{-\gamma t/2} \cos\left(\frac{\alpha t}{2}\right) + c_2 e^{-\gamma t/2} \sin\left(\frac{\alpha t}{2}\right)$$

that is a sine curve modulated by an exponential. The constants c_1 and c_2 are determined from the initial conditions. With the code `pend_re_soF` (real pendulum, small oscillations towards a focus) we can study the behaviour of the system both in phase space and in time domain.

```
# Real dumped pendulum: small oscillation (pend_re_soF).
# as Code 3.1 Ideal pendulum: small oscillations (pend_id_s_o.R),
# but here with the dumping constant gamma=0.25 (towards a focus)
# unstable focus: gamma<- -0.25, cinit<- c(0.001,0.001);
# see also Real dumped pendulum: small oscillation (pend_re_soN) (node).

parms<- c(1,2,0.25)
tinit<-0
tfin<-50
step<-0.01
times<-seq(tinit,tfin,by=step)
funct<- function(t,integ,p){ # the system of equations
x<-integ[1]
y<-integ[2]
dx<- parms[1]*y
dy<- -parms[3]*y-parms[2]*x
list(c(dx,dy))
} # end of funct
require(deSolve)
cinit<- c(2,3.5)
xy<-lsoda(cinit,times,funct,parms)
# to plot different trajectories, you have to choose:
# either in phase space or in time domain. With two runs you have both.

# phase space:
plot(xy[,2],xy[,3],type="l",xlab="x(t)",ylab="y(t)",xlim=c(-4,4),ylim=c(-4,4),
cex.lab=1.5,cex.axis=1.2,lwd=3,lty=1)
# time domain:
#plot(xy[,1],xy[,2],type="l",xlab="t",ylab="x(t)",xlim=c(0,50),ylim=c(-3,3),
#cex.lab=1.5,cex.axis=1.2,lwd=3,lty=1)
## second initial point
cinit<-c(2,0.2)
xy<- lsoda(cinit,times,funct,parms)
lines(xy[,2],xy[,3],col="red",lwd=3,lty=2) # phase space
#lines(xy[,1],xy[,2],col="red",lwd=3,lty=2) # time domain
```

Figure 4 shows the results. The trajectory of $x(t)$ as a function of time shows an oscillatory damping towards $x(t) = 0$. In the phase space the orbit are spirals and, since $\gamma > 0$, all the trajectories converge towards the origin $(x(0), y(0)) = (0, 0)$, a stable fixed point (*asymptotically stable*), called *focus* and, as we noted in Sect. 4.5, the basin of attraction is the entire (x, y) plane (see also Fig. 6.3 (a) and (b)). So that for every point $(x(t), y(t))$, it is:

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} y(t) = 0$$

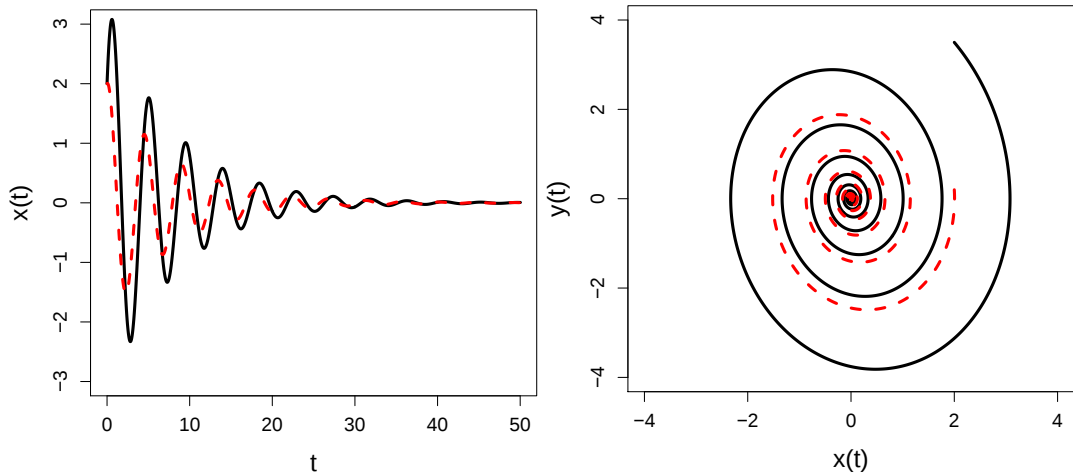


Figure 4 Time plot (left) and phase space portrait (right) of the time evolution of a damped pendulum with small oscillations. Solid and dashed lines refer to different initial conditions: $(2, 3.5)$ and $(2, 0.2)$.

Mathematically, we can put $\gamma < 0$, in this case the origin $(0, 0)$ is still a fixed point, but unstable (see Fig. 6.3 (c) and (d)) Lastly, if $\gamma = 0$, we return to the ideal pendulum and the origin is a centre.

Let us see now the second case, that is when: $\gamma^2 > 4\omega^2$.

Both the eigenvalues $\lambda_{1,2}$ are negative, the general solution is:

$$x(t) = c_1 \exp\left(\frac{-\gamma + \alpha}{2}\right)t + c_2 \exp\left(\frac{-\gamma - \alpha}{2}\right)t$$

In this case the motions are called overdamped. In the code `pend_re_sON` (real pendulum, small oscillations towards a node) we have chosen $\omega^2 = 2$, $\gamma = 3$, so $\lambda_1 = -1$, $\lambda_2 = -2$ and $\alpha = 1$. This code is as the previous code `pend_re_sOF` with different parameters and with four trajectories. It is reported below for the sake of completeness:

```
# Real damped pendulum: small oscillation (pend_re_sON).
# as Code 3.1 Ideal pendulum: small oscillations (pend_id_s_o.R),
# but here with the dumping constant gamma=3 (towards a node);
# see also Real damped pendulum: small oscillation (pend_re_sOF) (focus).

parms<- c(1,2,3)
tinit<-0
tfin<-10
step<-0.01
times<-seq(tinit,tfin,by=step)
funct<- function(t,integ,p){ # the system of equations
x<-integ[1]
y<-integ[2]
dx<- parms[1]*y
dy<- -parms[3]*y-parms[2]*x
list(c(dx,dy))
} # end of funct
require(deSolve)
cinit<- c(2.5,2.5)
xy<-lsoda(cinit,times,funct,parms)
```

```

# to plot different trajectories, you have to choose:
# either in phase space or in time domain. With two runs you have both.

# phase space:
plot(xy[,2],xy[,3],type="l",xlab="x(t)",ylab="y(t)",xlim=c(-3,3),ylim=c(-3,3),
     cex.lab=1.5,cex.axis=1.2,lwd=3,lty=1)
# time domain:
#plot(xy[,1],xy[,2],type="l",xlab="t",ylab="x(t)",xlim=c(0,10),ylim=c(-3,3),
#     #cex.lab=1.5,cex.axis=1.2,lwd=3,lty=1)
## second initial point
cinit<-c(-2.5,-2.5)
xy<- lsoda(cinit,times,func,parms)
lines(xy[,2],xy[,3],col="red",lwd=3,lty=2) # phase space
#lines(xy[,1],xy[,2],col="red",lwd=3,lty=2) # time domain

# further initial points
cinit<-c(1.5,1.5)
xy<- lsoda(cinit,times,func,parms)
lines(xy[,2],xy[,3],col="magenta",lwd=4,lty=3) # phase space
#lines(xy[,1],xy[,2],col="magenta",lwd=8,lty=3) # time domain

cinit<-c(-1.5,-1.5)
xy<- lsoda(cinit,times,func,parms)
lines(xy[,2],xy[,3],col="blue",lwd=3,lty=4) # phase space
#lines(xy[,1],xy[,2],col="blue",lwd=3,lty=4) # time domain

```

The results are shown in the following Fig. 5 In the time plot, we see that, for all the ini-

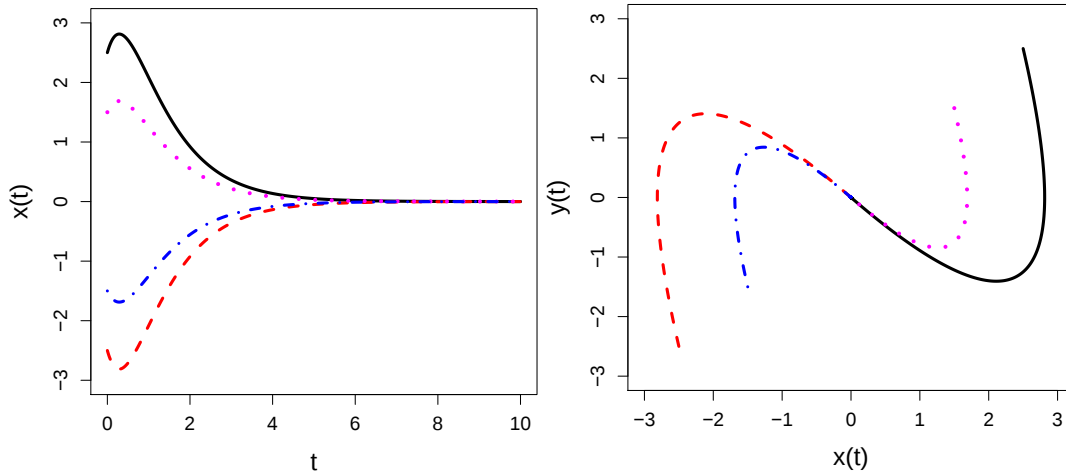


Figure 5 Time plot (left) and phase space portrait (right) of the time evolution of a damped pendulum with small oscillations. Different line types refer to different initial conditions.

tial conditions, trajectories converge to the origin in monotonic way. Imagine a pendulum immersed in a viscous liquid, the system accomplishes at the most half an oscillation, to get going sadly towards the origin and rest there. As we have seen in Sect. 6.2, the origin is a fixed stable point, called *node* (see Fig. 6.2 (a) and (b)). If both eigenvalues $\lambda_{1,2}$ are positive, then the origin will be an unstable fixed point, that is an unstable node (see Fig. 6.2 (c) and (d)).

Exercise: Study the dynamical behaviour of the damped pendulum with unlimited oscillations.

Solution: If we consider the possibility that the pendulum can perform also a rotation motion, in the general equation there is the term $\sin x$, that is:

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega^2 \sin x = 0$$

or equivalently:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\gamma y - \omega^2 \sin x \end{cases} \quad (7)$$

As for the ideal pendulum, in general, the fixed points are given by:

$$(x^*, y^*) = (n\pi, 0), \quad n = \dots, -2, -1, 0, +1, +2, \dots$$

Linearizing around the fixed point (x^*, y^*) , we find again the system:

$$\begin{cases} A = 0 \\ C = -\omega^2 \cos x|_{(x^*=n\pi)} = \begin{cases} -\omega^2 & (n \text{ even}) \\ +\omega^2 & (n \text{ odd}) \end{cases} \end{cases} \quad \begin{matrix} B = 1 \\ D = -\gamma \end{matrix}$$

The difference with the ideal pendulum is in the term $D = -\gamma$. The eigenvalue equation becomes:

$$\begin{aligned} \lambda^2 - (-\gamma)\lambda - (-\omega^2) &= 0 \quad (n \text{ even}) \\ \lambda^2 - (-\gamma)\lambda - (+\omega^2) &= 0 \quad (n \text{ odd}) \end{aligned}$$

For n even, λ_1 and λ_2 are complex and conjugated, we are in the case of the oscillatory motion towards the origin (focus). Also the fixed points

$$y^* = 0 \text{ e } x^* = \dots, -2\pi, 0, +2\pi, \dots$$

are foci. For n odd, λ_1 and λ_2 are real of opposite sign, then the fixed points

$$y^* = 0 \text{ e } x^* = \dots, -3\pi, -\pi, +\pi, +3\pi, \dots$$

are saddles.

Consider the following possible scenario. The pendulum starts with an energy enough to pass the unstable vertical point $\theta = \pi$ and so to establish a rotation motion. But unlike the ideal pendulum, the energy is dissipated by the frictional effects. The rotation velocity tends to become slower, until the system changes the quality of the motion, from rotation it becomes libration. Oscillations are dying and lastly the system finds its peace when the angle θ and the angular velocity $d\theta/dt$ are zero.

Figure 6 shows the features of the motion of the real pendulum, above described in words. The figure above is obtained with the code `pend_real`, derived from the Code 6.1, based on R package `phaser` (Grayling, 2015). Computation of the nullclines, the x and y variables as a function of time are left to the reader. The code is reported below.

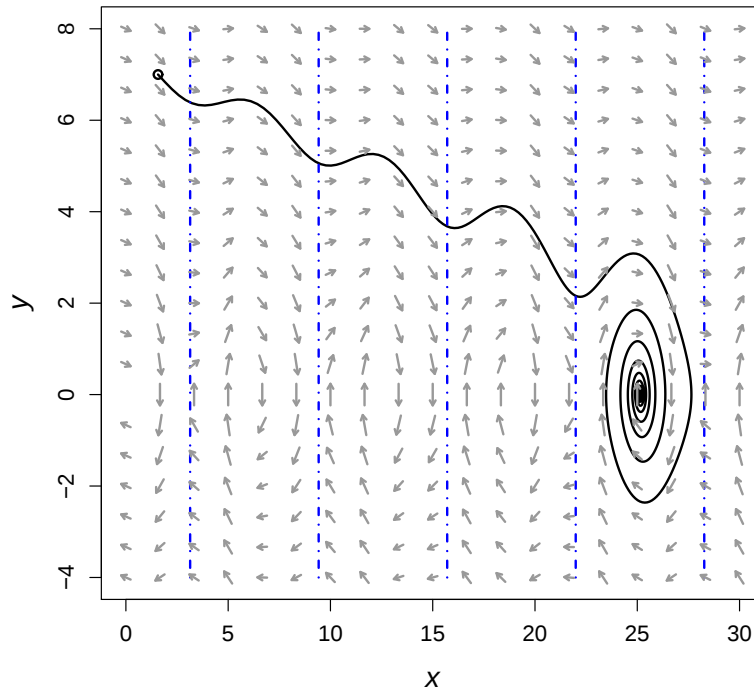


Figure 6 Dynamics behaviour in the phase space of a damped pendulum. Dash-dot lines cross the horizontal axis at saddle points, that is at $\pi, 3\pi, 5\pi, 7\pi$ and 9π . The velocity field is also plotted.

```
# Real damped pendulum: initial unlimited oscillations (pend_re_un_o)
library(phaseR)
#Set parameters for integration
t.end<- 50 #integrate from t=0 to t.end
delta<-0.05
pend<-function(t,y,parameters){
  x <- y[1]
  y <- y[2]
  a<-parameters[1]
  b<-parameters[2]
  c<-parameters[3]
  d<-parameters[4]
  dy<-numeric(2)
  dy[1]<-a*x+b*y
  dy[2]<-c*sin(x)+d*y
  list(dy)
}
omega2<- 2.
gamma<- 0.2
parms<-c(0,1,-omega2,-gamma) #parameter values
x0<- pi/2
y0<- 7
xy0<-cbind(x0,y0) #initial conditions
plot(0,0,type="n" ,xlim=c(0,30),ylim=c(-4,8),
  cex.axis=1.2,cex.lab=1.5,font.lab=3
  ,xlab="x",ylab="y")
flow<-trajectory(pend,y0=xy0,t.end=t.end,t.step=delta,parameters=parms,
  lwd=2,lty=1,colour="black",add=TRUE )
seq_pi<-seq(pi,+10*pi,by=2*pi)
```

```
seq_pi  
segments(seq_pi, -4, seq_pi, 8, col="blue", lwd=2, lty=4)  
arrows<-flowField(pend, lwd=2, x.lim=c(0, 30), y.lim=c(-4, 8),  
parameters=parms, points=19, arrow.head=0.05, frac=1, colour="gray60", add=TRUE)
```