

Exercise 4.2: Rossler attractor

In 1976 Otto Eberhard Rössler published the paper *An Equation for Continuous Chaos* (Physics Letters, 57A, 397-398). He modeled a chemical reaction mechanism with the system of three coupled differential equations:

$$\begin{aligned}dx/dt &= -(y + z) \\ dy/dt &= x + ay \\ dz/dt &= b + z(x - c)\end{aligned}$$

Note that the first two equations are linear, while the only nonlinear term is in the third equation, that is $z \times x$.

In the original article the parameters have the following values: $a = 0.20, b = 0.20, c = 5.70$. In the literature we find also other values, for which the system is chaotic, for instance: $a = 0.10, b = 0.10, c = 14$. In the package `tseriesChaos`, the function **rossler.syst** uses: $a = 0.15, b = 0.20, c = 10$, perhaps these values are the most commonly used, as we do in the following code. The code is reported below. Its structure is the same as the code `Lorenz attractor: Complete`, but some graphical parameters are changed.

Exercise: Study the Rössler system with the code below. The reader is also encouraged to explore what happens by varying the values of a, b, c .

Solution: Run the code `Rossler attractor: Complete`. The reader will realise how the choice of the parameters to estimate the MCLE and D_2 is delicate.

```
# Rossler attractor: Complete
library(tseriesChaos)
parms<- c(0.15,0.20,10) # parameters: a, b, c
tinit<- 0
tfin<- 650
step<- 0.1
times<- seq(tinit,tfin,by=step)
funct<- function(t,integ,parms) {
  x<-integ[1]
  y<-integ[2]
  z<-integ[3]
  a<- parms[1]
  b<- parms[2]
  c<- parms[3]
  dx<- -(y+z) # that is dx/dt = -(y+z)
  dy<- x+a*y # that is dy/dt = x+ay
  dz<- b+z*(x-c) # that is dz/dt = b+z(x-c)
  list(c(dx,dy,dz))
} # end of funct

require(deSolve)
cinit<-c(0,0,0)
xyz<-lsoda(cinit,times,funct,parms)
#xyz # comment if you do not wish the xyz values printed
par(mfrow=c(3,1))
par(mar = c(6.3, 4.8, 1., 3))
par(cex.lab=2,cex.axis=1.6,lwd=1,lty=1)
plot(xyz[,1],xyz[,2],type="l",xlab="t",ylab="x(t)", # x(t) vs t
```

```

xlim=c(tinit,tfin),ylim=c(-30,30))
plot(xyz[,1],xyz[,3],type="l",xlab="t",ylab="y(t)", # y(t) vs t
xlim=c(tinit,tfin),ylim=c(-30,30))
plot(xyz[,1],xyz[,4],type="l",xlab="t",ylab="z(t)", # z(t) vs t
xlim=c(tinit,tfin),ylim=c(0,50))
windows()
# phase space portrait
require(scatterplot3d)
scatterplot3d(xyz[,2],xyz[,3],xyz[,4],type="l",xlim=c(-20,20),
cex.lab=1.4,cex.axis=1.2)

windows()
par(mai=c(1.02,1.,0.82,0.42)+0.1)
plot(xyz[,2],xyz[,3],type="p",pch=20,cex=0.7,xlab="x(t)",ylab="y(t)",
cex.lab=1.5,cex.axis=1.2,lwd=3,lty=1,xlim=c(-1.5,1.5),ylim=c(-1,1))

trans<- 1000 # integration time step considered as transient
# discard initial transient:
x <- window(xyz[,2],trans)
y <- window(xyz[,3],trans)
z <- window(xyz[,4],trans)
t_start<- trans*step - step # new initial time
t_time<-seq(t_start,tfin,by=step) # new time interval
windows()
par(mfrow=c(3,1))
par(mar = c(6.3, 4.8, 1., 3))
par(cex.lab=2,cex.axis=1.6,lwd=1,lty=1)
# x(t), y(t), z(t) after the transient:
plot(t_time,x,type="l",xlab="t",ylab="x(t)",
xlim=c(t_start,tfin),ylim=c(-30,30))
plot(t_time,y,type="l",xlab="t",ylab="y(t)",
xlim=c(t_start,tfin),ylim=c(-30,30))
plot(t_time,z,type="l",xlab="t",ylab="z(t)",
xlim=c(t_start,tfin),ylim=c(0,50))

windows()
par(mai=c(1.02,1.,0.82,0.42)+0.1)
plot(x,y,type="p",pch=20,cex=0.7,xlab="x(t)",ylab="y(t)",
cex.lab=1.5,cex.axis=1.2,lwd=2,lty=1,xlim=c(-1.5,1.5),ylim=c(-1,1))

m.max<- 6 # embedding dimensions: from 1 to m_max
d<- 40 # tentative time delay (see below)
tw<- 100 # Theiler window
rt<- 10 # escape factor
eps<- sd(x)/10 # neighbourhood diameter
fn <- false.nearest(x,m.max,d,tw,rt,eps)
fn
windows()
plot(fn)

windows()
lm<- 60 # largest lag
mutual(x,lag.max = lm) # average mutual information to suggest d

```

```

# embedding Procedure
m<- 3 # choose embedding dimension
d<- 10 # choose time delay
xyz <- embedd(x,m,d) # embed the 'observed' series
windows()
scatterplot3d(xyz, type="l")

windows()
n<- 800
par(mai=c(1.02,1.,0.82,0.42)+0.1)
plot(0,0,type="n",xlab="i",ylab="j",xlim=c(1,n),ylim=c(1,n),cex.lab=1.6,cex.axis=1.2,
family="Times",font.lab=3)
distx<- matrix(,n,n)
eps<- 5
xx<- xyz[,1]
yy<- xyz[,2]
zz<- xyz[,3]
for(i in 1:n){ # construction of the thresholded recurrence plot
for(j in 1:n){
# Euclidean distance:
distx[i,j]<- sqrt( (xx[i]-xx[j])^2 + (yy[i]-yy[j])^2 + (zz[i]-zz[j])^2 )
if(distx[i,j]<eps) points(i,j,pch=19,cex=0.01)
}
}

# unthresholded recurrence plot

time<- seq(1:n)
windows()
par(mai=c(1.02,1.,0.82,0.42)+0.1,cex.lab=1.6,cex.axis=1.2)
require(gplots)
distx<- distx/max(distx)
distx<- 1-distx
filled.contour(time,time,distx,col=gray.colors(10,start=1,end=0),nlevels=10,
xlab = "i", ylab = "j", main = "",xlim=c(0,n),ylim=c(0,n),las=0,
key.axes = axis(4, las=1) )

# MCLE: maximum characteristic Lyapunov exponent (Kantz)

S_nu <- lyap_k(x,m=3,d=10,t=20,k=6,ref=5000,s=600,eps=5)
windows()
par(mai=c(1.02,1.,0.82,0.42)+0.1)
plot(S_nu,xlab = expression(paste(nu)),ylab=expression(paste("S", (nu))),
lwd=2,lty=1,cex=1.5)
lmin<- 80
lmax<- 280
# to add the vertical lines delimiting the linear region:
abline(v=lmin,lty=4,lwd=2,col="black")
abline(v=lmax,lty=4,lwd=2,col="black")
lr<- S_nu[lmin:lmax] # output of lyap_k(.) in [lmin, lmax]
l_lr<- length(lr)-1
lt<- seq(lmin*step,lmax*step,by=step)
lm(lr~lt) # regression analysis in R
abline(lyap(S_nu,lmin,lmax),lty=2,lwd=2,col="black")

```

```

# estimate of D2
C.m <- d2(x,m=6,d=10,t=100,eps.min=0.01,neps=100) # correlation integral, m = 1,...,
C.m <- data.frame(unclass(C.m)) # class attribute removed
C.3 <- subset(C.m,eps > 0.8 & eps < 6, select=c(eps,m3)) # eps in [0.8, 6]
lm(log(m3) ~ log(eps), data = C.3) # D2 estimate with m=3
windows() # if other plots are made before
par(mai=c(1.02,1.,0.82,0.42)+0.2)
plot(C.m[,1],C.m[,4], type="l",log="xy",main="",
xlab=expression(paste(epsilon)),ylab=expression(paste(widehat(C),(epsilon))),
lwd=3,lty=1,cex.axis=1.3,cex.lab=2.0, xlim=c(0.01,100),ylim=c(0.00000002,1))

# stop here to plot only C(m=3)
#####

# add the lines below to plot C(m=1,...,6)
n.m<- ncol(C.m)
for(i in (n.m-0):2) lines(C.m[,c(1,i)],lwd=2,lty=2)

```

The code yields the following figures:

1. Solutions of the Rössler equations.
2. Phase space portrait of the chaotic attractor.
3. Projection of the trajectory in the three-dimensional phase space onto the (x, y) plane
4. As the first figure, but without the transient.
5. Projection of the trajectory in the three-dimensional phase space onto the (x, y) plane without the transient.
6. Percentage of false nearest neighbours as a function of the embedding dimension, for the “observed” series x .
7. The average mutual information as a function of the lag, for the “observed” series x .
8. Plot in the reconstructed phase space of the Rössler strange attractor.
9. Thresholded recurrence plot.
10. Unthresholded recurrence plot.
11. Evolution of the logarithm of the mean distance $S(\nu)$ as a function of the time step ν .
12. Estimated correlation integral with the embedding dimension from $m = 1$ to $m = 6$.

The estimates of the invariants result to be $\hat{\lambda} = 0.075$ and $\hat{D}_2 = 1.90$. With the original parameters $a = 0.20, b = 0.20, c = 5.70$, the invariants are slightly different: $\hat{\lambda} = 0.071$ and $\hat{D}_2 = 1.82$

You can experiment the effect of the noise by adding the same instructions as we did in the code `lorenz attractor`. Recall that for the normal distribution the line is `x<-x + rnorm(length(x), 0, sd(x)/w)` (see p. 74), where `sd(x)` is the standard deviation of the series. You can simulate different noise model, for instance, based on the uniform distribution.