An Introduction to Forte's Set Theory

Contents

1 Historical Context

Pitch-class set theory emerged during the twentieth century as a response to the progressive dissolution of traditional tonal language in modern music. During the late nineteenth and early twentieth centuries, composers such as Claude Debussy, Alexander Scriabin, Arnold Schoenberg, Alban Berg, and Anton Webern increasingly explored harmonic structures that could no longer be fully explained through functional tonal theory.

As chromaticism intensified and tonal centers became unstable, traditional analytical methods based on major and minor harmony proved insufficient for the study of atonal and post-tonal music. In particular, the development of atonality and twelve-tone composition by Arnold Schoenberg and the Second Viennese School created the need for a new analytical framework capable of describing musical organization independently from tonal hierarchy.

Before the publication of Allen Forte's theoretical system, several theorists had already investigated intervallic relations, unordered pitch collections, and symmetrical structures in modern music. However, these approaches lacked a unified mathematical and classificatory methodology.

Allen Forte systematized these ideas in his influential book *The Structure of Atonal Music*, published in 1973. In this work, Forte introduced a rigorous method for the classification of pitch-class sets through intervallic content, transpositional equivalence, inversional equivalence, interval vectors, normal order, and prime form.

His theory transformed the analysis of post-tonal music by providing a formal language capable of describing musical structures independently from tonal function, register, instrumentation, or compositional style. Pitch-class set theory subsequently became one of the foundational analytical tools in twentieth-century music theory and remains central to the study of atonal, serial, and transformational musical systems.

2 Introduction

Pitch-class set theory is a method of musical analysis for the study of post-tonal and atonal music. Unlike traditional tonal theory, which is based on harmonic functions, hierarchical tonal centers, and voice-leading conventions, set theory focuses on the internal intervallic relationships between pitches independently of tonal context.

The central idea of the theory is that musical structures may be understood as collections of pitch classes related through intervallic organization and transformational operations. Rather than asking what key a chord belongs to, or what harmonic function it has, pitch-class set theory asks how the pitches are internally organized.

3 Pitch and Pitch Classes

In set theory, a distinction is made between the concepts of *pitch* and *pitch class*.

A *pitch* refers to a specific musical sound characterized by an exact frequency and register. For example, the pitches C_3, C_4, and C_5 represent different octave positions and therefore different physical pitches.

A *pitch class*, by contrast, is the collection of all pitches separated by one or more octaves and considered equivalent within the chromatic system. Thus, all occurrences of the note C belong to the same pitch class, regardless of octave.

Pitch-class theory abstracts away from octave placement and focuses only on the chromatic identity of a note. In twelve-tone equal temperament, pitch-class space is represented through modulo-12 arithmetic:

$$\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}.$$

Each integer corresponds to a chromatic pitch class:

Pitch Class	C	$C\sharp$	D	$E\flat$	E	F	$F\sharp$	G	$A\flat$	A	$B\flat$	B
Value	0	1	2	3	4	5	6	7	8	9	10	11

Under modulo-12 equivalence:

$$0 \equiv 12 \equiv 24 \pmod{12}.$$

For instance:

$$12 \equiv 0 \pmod{12}, \qquad 13 \equiv 1 \pmod{12}, \qquad 17 \equiv 5 \pmod{12}.$$

Thus, an octave corresponds to unison, a compound minor ninth corresponds to a minor second, and an eleventh corresponds to a perfect fourth.

For example:

$$C_2,\ C_3,\ C_4,\ C_5$$

are all represented by pitch class:

$$0.$$

Similarly:

$$G_2,\ G_3,\ G_4$$

all belong to pitch class:

$$7.$$

Pitch-class representation therefore abstracts musical material from register and octave placement, allowing musical structures to be analyzed according to their intervallic and transformational relationships rather than their absolute pitch height.

4 Pitch-Class Reference and Transpositional Equivalence

In Forte's standard pitch-class notation, pitch class 0 is conventionally associated with C:

$$0 = C.$$

However, this assignment is not an absolute musical necessity but rather a conventional analytical reference system. The numerical labels do not represent fixed pitches in an absolute sense; instead, they represent positions inside modulo-12 pitch-class space.

What remains structurally significant in set theory is not the absolute note name, but the intervallic relationship between pitch classes. For this reason, the reference point PC_0 may itself be transposed.

For example, one may redefine:

$$0 = A$$

or:

$$0 = F\sharp$$

while preserving exactly the same intervallic structure.

This follows from the principle of *transpositional equivalence*. A pitch-class set such as:

$$\{0, 4, 7\}$$

represents a major-triad structure when $0 = C$, yielding:

$$C\text{–}E\text{–}G.$$

But under transposition by two semitones:

$$T_2 : \{0, 4, 7\} \to \{2, 6, 9\},$$

it becomes:

$$D\text{–}F\sharp\text{–}A.$$

Although the absolute pitches have changed, the intervallic organization remains identical. Pitch-class theory therefore prioritizes structural intervallic relations over absolute pitch identity.

5 Interval Classes

In Forte's pitch-class set theory, musical intervals are grouped into *interval classes* (ICs). An interval class represents the shortest distance between two pitch classes, measured in semitones and considering inversional equivalence.

The interval class of a distance d is calculated as:

$$ic(d) = \min(d, 12 - d).$$

For example, the distance from 0 to 7 is seven semitones, but its interval class is:

$$ic(7) = \min(7, 12 - 7) = 5.$$

Thus, a perfect fifth and a perfect fourth both belong to $IC5$.

There are six interval classes:

Interval Class	Equivalent Intervals
IC1	minor second / major seventh
IC2	major second / minor seventh
IC3	minor third / major sixth
IC4	major third / minor sixth
IC5	perfect fourth / perfect fifth
IC6	tritone

For example:

- C–C# = IC1,
- C–D = IC2,
- C–E♭ = IC3,
- C–E = IC4,
- C–G = IC5,
- C–F# = IC6.

The interval-class system treats inversionally related intervals as equivalent. For instance, a minor second and a major seventh both belong to IC1.

6 Interval Vectors

An *interval vector* describes the intervallic content of a pitch-class set. It records how many times each interval class appears among all unordered pairs of pitch classes in the set.

The standard form of an interval vector is:

$$\langle ic1 \quad ic2 \quad ic3 \quad ic4 \quad ic5 \quad ic6 \rangle.$$

Each number indicates how many occurrences of each interval class appear inside the set.

6.1 Example 1: Major Triad

Consider the C major triad:

$$C \quad E \quad G.$$

In pitch-class notation this is:

$$\{0, 4, 7\}.$$

We examine all unordered pitch-class pairs:

1. $0 \leftrightarrow 4$,

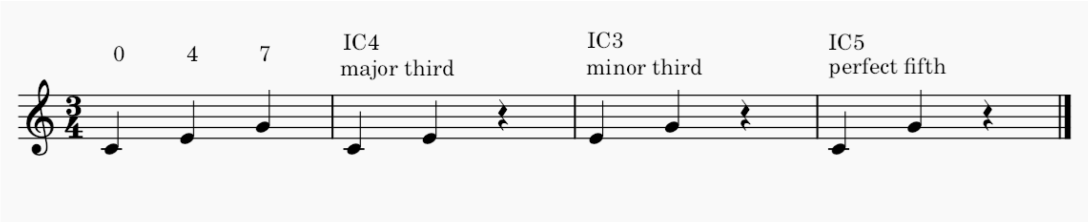

Figure 1: Example of interval classes in a major triad.

2. $4 \leftrightarrow 7$,
3. $0 \leftrightarrow 7$.

The interval classes are:

- $0 \leftrightarrow 4 = IC4$,
- $4 \leftrightarrow 7 = IC3$,
- $0 \leftrightarrow 7 = IC5$, since $ic(7) = 5$.

Counting the occurrences gives:

Interval Class	Occurrences
IC1	0
IC2	0
IC3	1
IC4	1
IC5	1
IC6	0

Therefore, the interval vector is:

$$\langle 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \rangle.$$

This vector shows that the major triad contains one minor third, one major third, and one perfect fourth/fifth.

6.2 Example 2: Diminished Triad

Consider the pitch collection:

$$C \quad E\flat \quad F\sharp.$$

In pitch-class notation this is:

$$\{0, 3, 6\}.$$

The interval classes are:

- $0 \leftrightarrow 3 = IC3$,

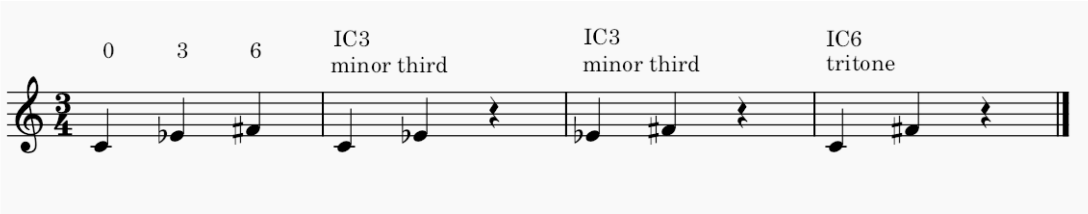

Figure 2: Example of interval classes in a diminished triad.

- $3 \leftrightarrow 6 = IC3$,
- $0 \leftrightarrow 6 = IC6$.

Counting the interval classes gives:

$$\langle 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 1 \rangle.$$

This interval vector highlights the characteristic structure of the diminished sonority: two minor thirds and one tritone.

Overall, an interval vector functions as a compact description of the internal intervallic structure of a pitch-class set. It does not indicate the order of notes, the register, or the specific pitches used. Instead, it describes only the distribution of interval classes within the set.

7 Normal Order and Prime Form

Different musical collections may belong to the same structural category even when they appear in different registral or transpositional forms. For this reason, pitch-class set theory introduces two fundamental procedures:

- **normal order**,
- **prime form**.

These procedures reduce pitch-class collections to canonical representations, allowing sets to be classified according to their internal intervallic structure.

7.1 Circular Pitch-Class Space

Pitch-class space is circular rather than linear. For this reason, when analyzing rotations of a set, it is often necessary to "unwrap" the circular structure into an ascending linear sequence.

For example:

$$(7, 0, 4)$$

cannot be directly measured as an ascending sequence because it crosses the octave boundary. Since:

$$0 \equiv 12 \pmod{12}$$

and:

$$4 \equiv 16 \pmod{12},$$

the ordered ascending realization becomes:

$$(7, 12, 16).$$

This transformation does not change the pitch classes themselves; it only preserves the ascending order required for measuring intervallic span.

7.2 Normal Order

The *normal order* of a pitch-class set is the cyclic arrangement of its elements that produces the smallest intervallic span between the first and the last pitch class when the set is written in ascending order.

Consider the pitch-class set:

$$\{0, 4, 7\}.$$

Its cyclic rotations are:

$$(0, 4, 7), \qquad (4, 7, 0), \qquad (7, 0, 4).$$

To compare their spans, rotations that cross the octave boundary must be unwrapped:

$$(4, 7, 0) \to (4, 7, 12),$$

because $0 \equiv 12 \pmod{12}$, and:

$$(7, 0, 4) \to (7, 12, 16),$$

because $0 \equiv 12 \pmod{12}$ and $4 \equiv 16 \pmod{12}$.

The spans are:

$$(0, 4, 7) \to 7 - 0 = 7,$$

$$(4, 7, 12) \to 12 - 4 = 8,$$

$$(7, 12, 16) \to 16 - 7 = 9.$$

The smallest span corresponds to:

$$(0, 4, 7),$$

which is therefore the normal order of the set.

7.3 The "Packed to the Left" Principle

In some cases, two or more cyclic rotations of a pitch-class set may produce the same total intervallic span between the first and the last element. When this occurs, the *packed to the left* principle is applied to determine the normal order.

The expression "packed to the left" means that preference is given to the ordering whose intervals are concentrated as closely as possible toward the beginning of the sequence. In practice, after comparing the total spans, the internal intervals are examined progressively from left to right.

For example, consider the two candidate orderings:

$$(0, 2, 3, 7)$$

and:

$$(0, 2, 5, 7).$$

Both structures span the same total intervallic distance:

$$7.$$

However, their internal intervallic distributions differ. The first ordering contains:

$$2,\ 1,\ 4,$$

while the second contains:

$$2,\ 3,\ 2.$$

Because the first structure places the smaller interval earlier in the sequence, it is considered more tightly packed toward the left. Consequently:

$$(0, 2, 3, 7)$$

is selected as the normal order.

7.4 Prime Form

The *prime form* is the most reduced and canonical representation of a pitch-class set under transposition and inversion.

The procedure is:

1. find the normal order;
2. transpose the set so that it begins with 0;
3. compute the inversion;
4. find the normal order of the inversion;
5. transpose the inversional form so that it begins with 0;
6. compare the two candidate forms;

7. choose the arrangement that is most compact toward the left.

Starting from the set:

$$\{0, 4, 7\},$$

the normal order is:

$$(0, 4, 7),$$

which already begins with 0.
Now compute the inversion:

$$(0, 8, 5).$$

Reordered in normal order, this becomes:

$$(0, 5, 8).$$

Since prime forms are defined up to transposition, the inversion may be transposed without changing its intervallic content. Transposing the inversion down by five semitones gives:

$$(0, 5, 8) - 5 = (7, 0, 3) \pmod{12},$$

which reordered becomes:

$$(0, 3, 7).$$

We now compare:

$$(0, 4, 7)$$

and:

$$(0, 3, 7).$$

Since $(0, 3, 7)$ is more compact toward the left, the prime form is:

$$(0, 3, 7).$$

This is the prime form associated with the major/minor triadic set-class. Because prime forms are defined up to transposition and inversion, sets belonging to the same intervallic structure may be transposed before comparison in order to obtain the most compact representative.

7.5 Normal Order and Prime Form: Conceptual Difference

Normal order and prime form serve different analytical purposes.
Normal order preserves:

- local spacing,
- registral compactness within the octave,

- intervallic distribution inside a specific cyclic arrangement.

Prime form preserves:

- abstract structural identity,
- transpositional equivalence,
- inversional equivalence.

Overall, normal order and prime form allow pitch-class sets to be reduced to canonical representations inside modulo-12 space.

8 Transformational Operations

Pitch-class set theory is based on two fundamental transformational operations: transposition and inversion.

8.1 Transposition

Transposition consists of shifting every pitch class in a set by the same intervallic amount within modulo-12 space. It is represented by:

$$T_n,$$

where n indicates the number of semitones by which the set is displaced.
The operation is formally defined as:

$$T_n(x) = x + n \pmod{12},$$

where x is a pitch class and n is the transposition index.
For example, applying T_2 to the C major triad gives:

$$T_2(\{0, 4, 7\}) = \{2, 6, 9\}.$$

The resulting pitch classes correspond to:

$$D\text{–}F\sharp\text{–}A.$$

Although the absolute pitches have changed, the intervallic structure remains identical.

8.2 Modulo-12 Transposition

Because pitch-class space is cyclic, all transpositions operate modulo 12. For example:

$$T_5(9) = 14,$$

and since:

$$14 \equiv 2 \pmod{12},$$

the result becomes:

$$T_5(9) = 2.$$

Thus, pitch classes always remain inside the chromatic space:

$$\{0, 1, 2, \ldots, 11\}.$$

8.3 Inversion

Inversion reflects pitch classes around a fixed axis inside modulo-12 space. In Forte's theory, inversion is represented by:

$$I_n,$$

and is formally defined as:

$$I_n(x) = n - x \pmod{12}.$$

For example, applying I_0 to the set $\{0, 4, 7\}$ gives:

$$I_0(\{0, 4, 7\}) = \{0, 8, 5\}.$$

Reordered into ascending form:

$$\{0, 5, 8\}.$$

This structure corresponds to the inversional form of the original major triad.

8.4 Transpositional and Inversional Equivalence

Two pitch-class sets are transpositionally equivalent if one can be obtained from the other by transposition. They are inversionally equivalent if one can be obtained from the other through inversion, possibly followed by transposition.

In Forte's theory, both transpositionally and inversionally equivalent sets belong to the same set-class because they preserve the same abstract intervallic structure.

9 Set-Classes and Forte Numbers

One of the central ideas of pitch-class theory is that different pitch-class collections may belong to the same *set-class*. A *set-class* is an abstract family of pitch-class sets related through transposition and inversion.

To organize and classify pitch-class sets, Forte developed a systematic catalog in which each set-class is assigned a unique label. The catalog groups together all sets that are equivalent under transpositional and inversional operations, thereby reducing multiple musical configurations to a single abstract category.

Each set-class label takes the form:

$$n\text{-}x,$$

where:

- n indicates the cardinality of the set, namely the number of pitch classes it contains;

- x indicates the position of the set-class within Forte's catalog for that cardinality.

For example:

$$3\text{-}11$$

designates the eleventh set-class among all tricords. The number 11 is not derived mathematically from the pitches themselves; it is the catalog index assigned by Forte to that particular tricordal set-class.

9.1 The Major/Minor Triad Set-Class

In tonal theory, major and minor triads are normally distinguished according to root, harmonic function, and modal quality. Pitch-class set theory, however, disregards these tonal attributes and classifies collections according to their intervallic content.

For example, the major triad:

$$(0, 4, 7)$$

and the minor triad:

$$(0, 3, 7)$$

are considered members of the same set-class. Although they possess different tonal functions, they contain the same intervallic structure through inversion.

The major triad contains the ordered intervallic succession:

$$4 + 3,$$

while the minor triad contains:

$$3 + 4.$$

Thus, the ordering of the intervals is reversed, but the unordered intervallic content remains invariant. After reduction to prime form, both the major and the minor triad correspond to:

$$(0, 3, 7).$$

In Forte's catalog of tricords, this prime form is assigned the label:

$$3\text{-}11.$$

Consequently, pitch-class set theory does not treat major and minor triads as fundamentally different structures, but rather as inversionally related manifestations of the same abstract intervallic configuration.

10 Catalog of Set-Classes

In *The Structure of Atonal Music* (1973), Allen Forte organized all pitch-class sets into a systematic catalog of set-classes. Each set-class groups together all pitch-class sets related through transposition and inversion.

The following table presents a selection of common dyadic, tricordal, and tetrachordal set-classes together with their corresponding prime forms.

Set-Class	Prime Form	Common Association
2-1	(0,1)	minor second
2-2	(0,2)	major second
2-3	(0,3)	minor third
2-4	(0,4)	major third
2-5	(0,5)	perfect fourth
2-6	(0,6)	tritone
3-1	(0,1,2)	chromatic cluster
3-6	(0,2,4)	whole-tone segment
3-9	(0,2,7)	quartal trichord
3-10	(0,3,6)	diminished triad
3-11	(0,3,7)	major/minor triad
3-12	(0,4,8)	augmented triad
4-1	(0,1,2,3)	chromatic tetrachord
4-13	(0,1,3,6)	all-interval tetrachord fragment
4-17	(0,3,4,7)	dominant seventh subset
4-21	(0,2,4,6)	whole-tone tetrachord
4-23	(0,2,5,7)	pentatonic fragment
4-27	(0,2,5,8)	half-diminished subset
4-28	(0,3,6,9)	diminished seventh chord
4-29	(0,1,3,7)	Z-related tetrachord

This catalog allows pitch-class sets to be compared independently of register, spacing, tonal center, or harmonic function. By reducing sets to their prime forms and assigning them to set-classes, Forte's system provides a unified method for analyzing atonal and post-tonal musical structures.

10.1 Hexachordal Set-Classes

Hexachords occupy a particularly important position within pitch-class set theory because of their relevance to serial and post-tonal music. Many hexachords exhibit strong symmetrical properties and are frequently employed in twelve-tone composition.

Set-Class	Prime Form	Common Association
6-1	(0,1,2,3,4,5)	chromatic hexachord
6-20	(0,1,4,5,8,9)	augmented collection fragment
6-27	(0,1,3,4,6,9)	all-trichord hexachord
6-30	(0,1,3,6,8,9)	combinatorial hexachord
6-32	(0,2,4,5,7,9)	diatonic hexachord
6-33	(0,2,3,5,7,9)	pentatonic collection
6-35	(0,2,4,6,8,10)	whole-tone collection
6-Z17	(0,1,2,4,7,8)	Z-related hexachord
6-Z19	(0,1,3,4,7,8)	Z-related hexachord
6-Z29	(0,1,3,6,7,9)	Z-related hexachord
6-Z44	(0,1,2,5,6,9)	Z-related hexachord
6-Z50	(0,1,4,6,7,9)	Z-related hexachord

Hexachords are especially significant because they often serve as structural partitions of the aggregate in twelve-tone music. Certain hexachords also possess high degrees of symmetry and combinatoriality, allowing them to generate complementary aggregate structures under transposition or inversion.

Particularly important is the whole-tone collection:

$$6\text{-}35,$$

whose prime form is:

$$(0, 2, 4, 6, 8, 10).$$

This set divides the octave into six equal whole-tone intervals and exhibits complete transpositional symmetry.

Another historically significant example is the all-trichord hexachord:

$$\text{6-27},$$

which contains representatives of all twelve trichordal set-classes within a single hexachordal structure.

11 Z-Relations

One of the most important and conceptually intriguing ideas in pitch-class set theory is the concept of the *Z-relation.* The letter "Z" is often associated with the idea of "twins": Z-related sets behave like intervallic twins because they share identical interval vectors while remaining structurally distinct.

Two pitch-class sets are said to be *Z-related* when they possess the same interval vector but are not related by either transposition or inversion.

To understand this idea, recall that the interval vector, abbreviated as IV, records the distribution of interval classes inside a set. If two sets possess the same interval vector, this means that they contain the same distribution of interval classes.

Formally, for two sets A and B:

$$IV(A) = IV(B).$$

However, sharing the same interval vector does not necessarily imply that the two sets belong to the same set-class. Normally, two pitch-class sets are considered equivalent if one can be transformed into the other by:

- transposition (T_n),
- inversion followed by transposition (T_nI).

Two sets are therefore Z-related when:

$$IV(A) = IV(B),$$

but at the same time:

$$A \not\sim T_n(B)$$

and:

$$A \not\sim T_nI(B)$$

for every possible value of n.

This means that no transposition or inversion can transform one set into the other, even though both sets contain exactly the same intervallic material.

11.1 Classical Example

One of the best-known Z-related pairs in Forte's catalog is formed by the set-classes:

$$4\text{-}Z15$$

and:

$$4\text{-}Z29.$$

These are represented respectively by the pitch-class sets:

$$\{0, 1, 4, 6\}$$

and:

$$\{0, 1, 3, 7\}.$$

At first sight the two sets appear different, and indeed no transposition or inversion can transform one into the other. For this reason, they belong to different set-classes. However, both sets possess exactly the same interval vector:

$$\langle 1\ 1\ 1\ 1\ 1\ 1 \rangle.$$

For the set:

$$\{0, 1, 4, 6\},$$

the interval-class relations are:

$$\begin{aligned} 0 \leftrightarrow 1 &= IC1, \\ 0 \leftrightarrow 4 &= IC4, \\ 0 \leftrightarrow 6 &= IC6, \\ 1 \leftrightarrow 4 &= IC3, \\ 1 \leftrightarrow 6 &= IC5, \\ 4 \leftrightarrow 6 &= IC2. \end{aligned}$$

Thus the set contains exactly one occurrence of every interval class from IC1 to IC6.
For the set:

$$\{0, 1, 3, 7\},$$

the interval-class relations are:

$$\begin{aligned} 0 \leftrightarrow 1 &= IC1, \\ 0 \leftrightarrow 3 &= IC3, \\ 0 \leftrightarrow 7 &= IC5, \\ 1 \leftrightarrow 3 &= IC2, \\ 1 \leftrightarrow 7 &= IC6, \\ 3 \leftrightarrow 7 &= IC4. \end{aligned}$$

Again, every interval class from IC1 to IC6 appears exactly once. The two sets therefore contain identical intervallic content, even though their internal organization differs and they cannot be transformed into one another through transposition or inversion. This exceptional situation is called a *Z-relation*.

11.2 Analytical Significance

Z-relations reveal one of the deepest aspects of Forte's theory:

- interval vectors describe intervallic content,
- but they do not completely determine structural identity.

Consequently, two sets may appear intervallically identical at the statistical level while remaining transformationally unrelated.

From a mathematical perspective, Z-related sets demonstrate that the mapping:

$$\text{Pitch-Class Set} \to \text{Interval Vector}$$

is not injective. Different pitch-class sets may therefore generate the same interval-vector signature.

Z-relations thus expose the limits of interval-vector analysis and motivate the need for additional structural descriptors such as normal order, prime form, transformational relations, and symmetry analysis.

12 Symmetry and Invariance

Many pitch-class sets possess internal symmetries generated through transposition or inversion. A set is considered symmetrical when one or more transformational operations map the set onto itself.

12.1 Transpositional Symmetry

Consider the augmented triad:

$$\{0, 4, 8\}.$$

Applying:

$$T_4$$

produces:

$$\{4, 8, 0\},$$

which is equivalent to the original set. Thus the structure is invariant under:

$$T_4.$$

This symmetry results from the equal distribution of intervals inside the set.

12.2 Diminished Symmetry

Similarly, the diminished seventh chord:

$$\{0, 3, 6, 9\}$$

is invariant under:

$$T_3,$$

because:

$$T_3(\{0, 3, 6, 9\}) = \{3, 6, 9, 0\},$$

which reproduces the same pitch collection.

12.3 Inversional Symmetry

Some pitch-class sets also possess inversional symmetry. This occurs when inversion around a specific axis reproduces the original set.

Symmetrical sets are particularly important because they often generate cyclic structures, intervallic regularity, invariant subsets, and transformational balance. For this reason, symmetry plays a central role in twentieth-century harmonic organization and transformational music theory.

13 Orbit Theory

Transformational operations generate cyclic trajectories inside pitch-class space known as *orbits*. An orbit is the ordered collection of pitch classes produced through repeated application of a transformational operator.

13.1 Example of a Transpositional Orbit

Consider the operation:

$$T_3$$

starting from pitch class:

$$0.$$

Repeated application produces:

$$0 \to 3 \to 6 \to 9 \to 0.$$

The orbit generated by T_3 is therefore:

$$\{0, 3, 6, 9\}.$$

Because:

$$4 \times 3 = 12 \equiv 0 \pmod{12},$$

the cycle closes after four iterations.

13.2 Periodicity

The length of an orbit depends on the periodicity of the transformational operation inside modulo-12 space. In general:

$$N = \frac{12}{\gcd(12, n)},$$

where n is the intervallic displacement and N is the orbit length.

13.3 Structural Interpretation

Orbit theory reveals the dynamic aspect of pitch-class organization. Instead of analyzing pitch collections as static objects, orbit analysis describes the transformational processes capable of generating cyclic musical structures.

14 Ordered and Unordered Pitch-Class Sets

One of the most important distinctions in post-tonal theory is the difference between ordered and unordered pitch collections.

Traditional pitch-class set theory primarily studies unordered pitch-class sets, in which the order of the elements is irrelevant. For example:

$$\{0, 4, 7\}$$

and:

$$\{7, 0, 4\}$$

represent the same unordered set.

In this context, analysis focuses on intervallic content, interval vectors, normal order, and prime form.

14.1 Ordered Structures

In ordered systems, however, the sequence of intervals becomes structurally significant. For example:

$$(4, 3, 5)$$

and:

$$(5, 3, 4)$$

generate different interval trajectories despite containing the same interval components.

Ordered interval systems therefore describe directional processes, intervallic succession, cyclic generation, and transformational evolution.

14.2 Static and Dynamic Perspectives

This distinction may be interpreted as the difference between:

static set theory

and:

dynamic generative theory.

Forte's theory primarily analyzes the internal structure of completed pitch-class sets, while ordered interval systems analyze the processes through which those structures are generated.

15 Applications of Pitch-Class Set Theory in Musical Analysis

Since the publication of Allen Forte's *The Structure of Atonal Music* (1973), pitch-class set theory has become one of the principal analytical tools for the study of twentieth-century and post-tonal repertoire.

The theory has been widely applied to compositions characterized by atonality, chromatic saturation, intervallic organization, symmetrical structures, and non-functional harmonic systems.

Among the composers most frequently analyzed through set theory are Arnold Schoenberg, Anton Webern, Alban Berg, Béla Bartók, Igor Stravinsky, Olivier Messiaen, Milton Babbitt, and Elliott Carter.

15.1 Arnold Schoenberg

Schoenberg's atonal and twelve-tone works are among the foundational examples of pitch-class analysis. Particularly important are *Three Piano Pieces*, Op. 11, *Pierrot Lunaire*, and the *Suite for Piano*, Op. 25.

These works exhibit highly organized intervallic structures and motivic cells that may be analyzed through pitch-class sets, interval vectors, and transformational operations.

15.2 Anton Webern

Webern's music is especially suited to set-theoretical analysis because of its extreme motivic concentration and symmetrical intervallic organization. Works frequently analyzed include the *Symphony*, Op. 21, *Variations for Piano*, Op. 27, and the *Five Movements for String Quartet*, Op. 5.

15.3 Alban Berg

Although Berg often preserves expressive and tonal references, his music contains complex post-tonal structures that may be examined through set theory. Examples include the opera *Wozzeck*, the *Lyric Suite*, and the *Violin Concerto.*

15.4 Béla Bartók

Bartók's music has often been analyzed through intervallic and symmetrical models closely related to pitch-class theory. Important examples include *Music for Strings, Percussion and Celesta*, the *Mikrokosmos*, and the *String Quartets*.

15.5 Igor Stravinsky

Stravinsky's harmonic language frequently employs repeating intervallic cells and non-functional pitch organizations. Set-theoretical analysis has been widely applied to *The Rite of Spring*, *Petrushka*, and *Symphonies of Wind Instruments*.

15.6 Milton Babbitt

Milton Babbitt extended serial techniques through highly systematic combinatorial and transformational procedures. His works are central to advanced pitch-class and transformational analysis, especially *Three Compositions for Piano*, *Composition for Four Instruments*, and *Philomel*.

15.7 Analytical Importance

Pitch-class set theory became particularly influential because it provided a formal analytical language capable of describing music independently from tonal function.

The theory allows analysts to identify recurring intervallic cells, transformational relations, symmetrical organizations, cyclic structures, and hidden structural coherence within highly chromatic and post-tonal musical environments.

For this reason, set theory remains one of the central analytical methodologies for twentieth-century music and continues to influence contemporary approaches to transformational and generative musical systems.